INTRODUCTION TO DG-CATEGORIES

k - a commutative ring

Defn 1: A **DG-category** $\mathcal{A}$ over k is a category whose

- Hom-spaces are complexes of k-modules: $\mathrm{Hom}^\bullet_{\mathcal{A}}(a,b) = \ldots \xrightarrow{d} \mathrm{Hom}^{i-1}_{\mathcal{A}}(a,b) \xrightarrow{d} \mathrm{Hom}^i_{\mathcal{A}}(a,b) \xrightarrow{d} \ldots$
- composition maps are degree 0 maps of complexes.

$$\forall i,j \in \mathbb{Z} \quad \underset{\underset{g}{\Psi}}{\mathrm{Hom}^i_{\mathcal{A}}(b,c)} \otimes_k \underset{\underset{f}{\Psi}}{\mathrm{Hom}^j_{\mathcal{A}}(a,b)} \longrightarrow \underset{\underset{(g\circ f)}{\Psi}}{\mathrm{Hom}^{i+j}_{\mathcal{A}}(a,c)} \quad \text{s.t. } d(g\circ f) = dg\circ f + (-1)^i g\circ df$$

Example 2: 1) Let R be a k-algebra. We can consider R as a DG-category:

$\underset{\text{single object}}{\mathrm{Obj}\, R = \{\bullet\}}$ and $\mathrm{Hom}^i_R(\bullet,\bullet) = \begin{cases} R & i=0 \\ 0 & i\neq 0 \end{cases} \longleftarrow R$ concentrated in deg. 0

2). More generally, let A be DG k-algebra. $\rightsquigarrow \mathrm{Ob}\, A = \{\bullet\}$, $\mathrm{Hom}^\bullet_A(\bullet,\bullet) = A$.

3). Let $\mathcal{B}$ be an abelian k-category. Define DG category **$C(\mathcal{B})$** by:

$\mathrm{Obj}\, C(\mathcal{B}) = \{$complexes of objects in $\mathcal{B}\}$

$\forall a^\bullet, b^\bullet \in C(\mathcal{B}) \quad \mathrm{Hom}^i_{C(\mathcal{B})}(a^\bullet, b^\bullet) = \{$deg i maps of underlying graded objects$\} = \coprod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(a^j, b^{i+j})$

with differential $d\alpha = d\circ\alpha - (-1)^{\deg\alpha}\alpha\circ d$.

NB: closed $\Leftrightarrow$ map of complexes (comm./anti-comm. with d)

For a k-algebra R write $C(R)$ for $C(\mathrm{Mod}\text{-}R)$. E.g. $C(k)$ is the DG category of complexes of k-modules.

For a scheme X over $\mathrm{Spec}\, k$ write $C(X)$ for $C(\mathcal{O}_X\text{-mod})$.

Defn 3: Let $\mathcal{A}$ be a DG-category. The **homotopy category** $H^0(\mathcal{A})$ is the (ordinary) category:

$\mathrm{Obj}\, H^0(\mathcal{A}) = \mathrm{Obj}(\mathcal{A}) \quad \mathrm{Hom}_{H^0(\mathcal{A})}(a,b) = H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(a,b))$.

E.g.: Let $\mathcal{A}$ be an abelian category.

$H^0(C(\mathcal{A}))$ = the category of complexes in $\mathcal{A}$ modulo homotopy $\leftsquigarrow K(\mathcal{A})$

$$\begin{tikzcd} \cdots \arrow[r, "d"] & a_{i-1} \arrow[r, "d"] \arrow[d] \arrow[dl] & a_i \arrow[r, "d"] \arrow[d] \arrow[dl] & a_{i+1} \arrow[r, "d"] \arrow[d] \arrow[dl] & \cdots \arrow[dl] \\ \cdots \arrow[r, "d"] & b_{i-1} \arrow[r, "d"] & b_i \arrow[r, "d"] & b_{i+1} \arrow[r, "d"] & \cdots \end{tikzcd}$$

$\ker d \subseteq \mathrm{Hom}^0_{C(\mathcal{A})}(a,b) \longleftrightarrow$ maps of complexes $a^\bullet \to b^\bullet$

$\cup\!\!|$

$\mathrm{im}\, d \subseteq \mathrm{Hom}^0_{C(\mathcal{A})}(a,b) \longleftarrow$ null homotopic maps

Defn 4: Let $\mathcal{A}, \mathcal{B}$ be DG-categories. A **DG-functor** $\mathcal{A} \xrightarrow{\Phi} \mathcal{B}$ is a functor such that

$\forall a,b \in \mathcal{A} \quad \mathrm{Hom}_{\mathcal{A}}(a,b) \to \mathrm{Hom}_{\mathcal{B}}(\Phi a, \Phi b)$ is a map of complexes of k-modules.

Define the DG-category **$\mathrm{DGFun}(\mathcal{A},\mathcal{B})$** by

$\mathrm{Ob}\, \mathrm{DGFun}(\mathcal{A},\mathcal{B}) = \{$DG-functors $\mathcal{A} \to \mathcal{B}\}$

$\mathrm{Hom}^i_{\mathrm{DGFun}(\mathcal{A},\mathcal{B})}(\Phi,\Psi) = \{$degree i natural transformations $\Phi \to \Psi\}$

$$\begin{tikzcd} \mathcal{A} & \Phi(a) \arrow[d, "\deg i"] \\ a \arrow[u, phantom, "\Psi" rotate=180] \arrow[ur, dotted] \arrow[r, dotted] & \Psi(a) \end{tikzcd}$$

Compositions and the differential are defined levelwise in $\mathcal{B}$, e.g. given $\Phi \xrightarrow{\alpha} \Psi$ define $\forall a \in \mathcal{A}$ $\Phi(a) \xrightarrow{d\alpha} \Psi(a)$ to be $\underset{\llcorner \text{ diff. in } \mathcal{B}}{d}\left(\Phi(a) \xrightarrow{\alpha} \Psi(a)\right)$.

by way of example

Defn 5: Let $\mathcal{A}$ be a DG-category. A *(right) $\mathcal{A}$-module* is a DG-functor $\mathcal{A}^{op} \longrightarrow C(k)$. These form a DG-category $DGFun(\mathcal{A}^{op}, C(k))$ which we denote by *$Mod(\mathcal{A})$*.

Examples 6: a) Let R be a k-algebra. Then $Mod\text{-}R_{DG} = C(R)$, the category of complexes of R-modules.

$$\begin{tikzcd} \bullet \arrow[loop left, "s\in R"] \arrow[r, squiggly, "M"] & M(\bullet) = \cdots \arrow[r] & M_{i-1} \arrow[r] \arrow[d, "M(s)"] & M_i \arrow[r] \arrow[d, "M(s)"] & M_{i+1} \arrow[r] \arrow[d, "M(s)"] & \cdots \\ & \cdots \arrow[r] & M_{i-1} \arrow[r] & M_i \arrow[r] & M_{i+1} \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} & \cdots \arrow[r] & M_{i-1} \arrow[r] \arrow[ddl] & M_i \arrow[r] \arrow[ddl] & M_{i+1} \arrow[r] \arrow[ddl] & \cdots \arrow[ddl] \\ \bullet \arrow[ur, squiggly, "M"] \arrow[dr, squiggly, "N"'] & & & & & \\ & \cdots \arrow[r] & N_{i-1} \arrow[r] & N_i \arrow[r] & N_{i+1} \arrow[r] & \cdots \end{tikzcd}$$

deg -1 morphism

b) Let A be a DG k-algebra. Then $Mod\text{-}A$ is the DG-category of (usual) DG-modules over A.

$$\begin{tikzcd} \bullet \arrow[loop left, "s\in A^j"] \arrow[r, squiggly, "M"] & \cdots \arrow[r] & M_{i-1} \arrow[r] \arrow[d, "s"] & M_i \arrow[r, "d_i"] \arrow[d, "s"] & M_{i+1} \arrow[r] \arrow[d, "s"] & \cdots \\ & \cdots \arrow[r] & M_{i+j-1} \arrow[r] & M_{i+j} \arrow[r, "d_{i+j}"'] & M_{i+j+1} \arrow[r] & \cdots \end{tikzcd}$$

$$ds.m = s.(dm) - (-1)^j d(s.m)$$

Some notation: Let $\mathcal{A}$ be a DG-category.

- $\forall E, F \in Mod(\mathcal{A})$ write $Hom_{\mathcal{A}}(E,F)$ for $Hom_{Mod(\mathcal{A})}(E,F)$.
- $\forall E \in Mod(\mathcal{A})$, $a \in \mathcal{A}$ write E_a for $E(a) \in C(k)$
- $\forall E \in Mod(\mathcal{A})$ write $s \in E$ for "$s \in E_a$ for some $a \in \mathcal{A}$".

Definition 7: The *Yoneda embedding* $\mathcal{A} \hookrightarrow Mod(\mathcal{A})$ is the fully faithful functor defined by:

$$\forall a \in \mathcal{A} \quad a \mapsto Hom_{\mathcal{A}}(-,a) \quad \forall \alpha \in Hom_{\mathcal{A}}(b,c) \quad \alpha \mapsto Hom_{\mathcal{A}}(-,b) \xrightarrow{\alpha\circ(-)} Hom_{\mathcal{A}}(-,c)$$

Some notation (ct-d):

- $\forall a \in \mathcal{A}$ write ${}_a\mathcal{A} \in Mod(\mathcal{A})$ for its image under Yoneda. These are *representable* $\mathcal{A}$-modules. NB: ${}_a\mathcal{A}_b = Hom_{\mathcal{A}}(b,a)$
- $\forall E \in Mod(\mathcal{A})$, $s \in E_a$, $\alpha \in {}_a\mathcal{A}_b$ write $\alpha.s \in E_b$ for $(-1)^{\deg\alpha \deg s} E(\alpha)(s)$.

We have by functoriality:

$$(s\cdot\alpha)\cdot\beta = s\cdot(\alpha\circ\beta)$$

Think of data defining E as a collection of fibers $E_a \in C(k)$ with a right action of $\mathcal{A}$ where $\alpha \in {}_a\mathcal{A}_b$ acts on E_a & sends it to E_b.

Similarly, a *left $\mathcal{A}$-module* is an $\mathcal{A}^{op}$-module, i.e. DG-functor $\mathcal{A} \to C(k)$.

$\forall E \in Mod\text{-}\mathcal{A}^{op}$ write ${}_aE \in C(k)$ for $E(a)$ $\forall a \in \mathcal{A}$

$\alpha.s \in E_a$ $\forall \alpha \in {}_a\mathcal{A}_b$, $s \in {}_bE$

Definition 8: Let $E \in Mod\text{-}\mathcal{A}$, $F \in Mod\text{-}\mathcal{A}^{op}$ define the *tensor product* *$E \otimes_{\mathcal{A}} F$* $\in C(k)$ as

$$\bigoplus_{a\in\mathcal{A}} E_a \otimes_k {}_aF \text{ modulo relations } (s.\alpha)\otimes t = s\otimes(t.\alpha) \quad \forall \alpha \in {}_a\mathcal{A}_b, s \in E_a, t \in F_b$$

OBSERVATION: FOR ANY DG-CATEGORY $\mathcal{A}$ THE HOMOTOPY CATEGORY $H^0(\mathrm{Mod}\text{-}\mathcal{A})$ IS NATURALLY TRIANGULATED. ⟵ DEFINED LEVELWISE IN $H^0(C(k)) = K(k)$ VIA ITS NATURAL TRIANGULATED STRUCT.

DEFINITION 9: AN $\mathcal{A}$-MODULE E IS CALLED

1) ACYCLIC IF E_a IS AN ACYCLIC COMPLEX $\forall a \in \mathcal{A}$.
2) h-PROJECTIVE IF $\mathrm{Hom}_{H^0(\mathrm{Mod}\mathcal{A})}(E, C) = 0$ $\forall$ ACYCLIC $C \in \mathrm{Mod}(\mathcal{A})$

   DENOTE BY $P(\mathcal{A})$ THE CORRESPONDING FULL SUBCATEGORY OF $\mathrm{Mod}(\mathcal{A})$.
3) SEMI-FREE IF $\exists$ A FILTRATION (LEVELWISE IN $C(k)$) $E_0 \subseteq E_1 \subseteq \ldots \subseteq E_i \subseteq \ldots$ OF E WHOSE QUOTIENTS ARE SHIFTS OF FREE REPRESENTABLE MODULES.

   E IS FURTHER CALLED FINITELY GENERATED IF $E_n = E_{n+1} = E_{n+2} = \ldots = E$ FOR $n \gg 0$.

   $SF(\mathcal{A})$, $SF_{fg}(\mathcal{A})$ - CORRESPONDING FULL SUBCATEGORIES OF $\mathrm{Mod}(\mathcal{A})$

A MORPHISM $E \xrightarrow{\alpha} F$ IS A QUASI-ISOMORPHISM IF $E_a \xrightarrow{\alpha} F_a$ IS A QUASI-ISO $\forall a \in \mathcal{A}$.

EXERCISE: a) PROVE THAT REPRESENTABLE MODULES ARE h-PROJECTIVE. BONUS: SEMI-FREE

b) $\forall E \in \mathrm{Mod}(\mathcal{A})$ TFAE:

1) E IS h-PROJECTIVE
2) $\forall$ DIAGRAM IN $H^0(\mathrm{Mod}(\mathcal{A}))$ WITH β A QUASI-ISO.

$$\begin{tikzcd} & F \arrow[d, "\beta"] \\ E \arrow[ur, dashed, "\exists ! \gamma"] \arrow[r, "\alpha"] & G \end{tikzcd}$$

3) EVERY QUASI-ISOMORPHISM $F \to E$ HAS A LEFT INVERSE IN $H^0(\mathrm{Mod}\mathcal{A})$.

c) DEFINE h-INJECTIVE MODULES & RESTATE a) FOR THEM.

DEFINITION 10: THE DERIVED CATEGORY $D(\mathcal{A})$ OF A DG-CATEGORY $\mathcal{A}$ IS THE LOCALISATION OF $H^0(\mathrm{Mod}\mathcal{A})$ BY QUASI-ISOMORPHISMS.

DEFINITION OF h-PROJECTIVITY $\Rightarrow$ NATURAL FUNCTOR $H^0(P(\mathcal{A})) \to D(\mathcal{A})$ IS FULLY FAITHFUL.

(DRINFELD) AN $\mathcal{A}$-MODULE ADMITS A RESOLUTION (QUASI-ISO) BY A SEMI-FREE MODULE.

k IS A FIELD $\Rightarrow$ THERE IS A FUNCTORIAL SEMI-FREE RESOLUTION:

$$\begin{tikzcd} \mathrm{Mod}\mathcal{A} \arrow[rr, "Q"] \arrow[dr] & & \mathrm{Mod}\mathcal{A} \\ & SF\mathcal{A} \arrow[ur, hook] & \end{tikzcd}$$

& $\mathrm{id} \xrightarrow{q.i.} Q$.

PROP. 11: $H^0(P(\mathcal{A})) \hookrightarrow D(\mathcal{A})$ IS AN EQUIVALENCE OF CATEGORIES.

ALSO: DRINFELD QUOTIENT $\mathrm{Mod}\mathcal{A}/\mathrm{ac}\mathcal{A} \rightsquigarrow H^0(\mathrm{Mod}\mathcal{A}/\mathrm{ac}\mathcal{A}) \simeq D(\mathcal{A})$

DEFINITION 12: LET $\mathcal{A}, \mathcal{B}$ BE DG CATEGORIES. THE TENSOR PRODUCT $\mathcal{A} \otimes \mathcal{B}$ IS THE DG-CATEGORY

OBJECTS: PAIRS (a,b) $a \in \mathcal{A}, b \in \mathcal{B}$ MORPHISMS: $\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}((a,b),(a',b')) = \mathrm{Hom}_{\mathcal{A}}(a,a') \otimes_k \mathrm{Hom}_{\mathcal{B}}(b,b')$.

AN $\mathcal{A}$-$\mathcal{B}$ BIMODULE IS A MODULE FOR $\mathcal{A}^{op} \otimes \mathcal{B}$. WRITE $\mathrm{Mod}\text{-}\mathcal{A}\text{-}\mathcal{B}$ FOR THE CORRESPONDING CATEGORY.

LET $M \in \mathrm{Mod}(\mathcal{A}\text{-}\mathcal{B})$. WRITE ${}_aM_b$ FOR $M(a,b) \in C(k)$. WRITE ${}_aM$ FOR $\mathcal{B}$-MODULE $M(a,-)$ & M_b FOR LEFT $\mathcal{A}$-MODULE $M(-,b)$.

Let $M \in \mathrm{Mod}(\mathcal{A}\text{-}\mathcal{B})$. The functor $\mathcal{A} \to \mathrm{Mod}\,\mathcal{B}$ $a \mapsto {}_aM$ extends to

$$(-)\otimes_{\mathcal{A}} M : \mathrm{Mod}\,\mathcal{A} \to \mathrm{Mod}\,\mathcal{B}$$

defined fiberwise over $\mathcal{A}$: $(E \otimes_{\mathcal{A}} M)_b = E \otimes M_b \in C(k)$

For any DG-category $\mathcal{C}$ $\rightsquigarrow$ $(-)\otimes_{\mathcal{A}}(-) : \mathrm{Mod}\,\mathcal{C}\text{-}\mathcal{A} \times \mathrm{Mod}\,\mathcal{A}\text{-}\mathcal{B} \longrightarrow \mathrm{Mod}\,\mathcal{C}\text{-}\mathcal{B}$

$\mathrm{Hom}_{\mathcal{A}}(-,-) : \mathrm{Mod}\,\mathcal{C}\text{-}\mathcal{A} \times \mathrm{Mod}\,\mathcal{B}\text{-}\mathcal{A} \longrightarrow \mathrm{Mod}\,\mathcal{B}\text{-}\mathcal{C}$

fiberwise over $\mathcal{B}, \mathcal{C}$ via $\mathrm{Hom}_{\mathcal{A}}(-,-) : \mathrm{Mod}\,\mathcal{A} \times \mathrm{Mod}\,\mathcal{A} \to C(k)$

Moreover,

$$(-)\otimes_{\mathcal{A}} M : \mathrm{Mod}\,\mathcal{C}\text{-}\mathcal{A} \to \mathrm{Mod}\,\mathcal{C}\text{-}\mathcal{B}$$

is **left adjoint** to

$$\mathrm{Hom}_{\mathcal{B}}(M,-) : \mathrm{Mod}\,\mathcal{C}\text{-}\mathcal{B} \to \mathrm{Mod}\,\mathcal{C}\text{-}\mathcal{A}.$$

Denote by $(-)\overset{L}{\otimes}_{\mathcal{A}} M$ & $R\mathrm{Hom}_{\mathcal{B}}(M,-)$ the corresponding (left and right) derived functors.

Definition 13: Let $M \in \mathrm{Mod}(\mathcal{A}\text{-}\mathcal{B})$. We say that

- M is **$\mathcal{A}$-h-projective** if $\forall b \in \mathcal{B}$ M_b is an h-projective $\mathcal{A}^{\mathrm{opp}}$-module.
- M is **$\mathcal{B}$-h-projective** if $\forall a \in \mathcal{A}$ ${}_aM$ is an h-projective $\mathcal{A}$-module.

Similarly for $\mathcal{A}$- and $\mathcal{B}$-acyclic, etc.

Exercise 14: Let $M \in \mathrm{Mod}(\mathcal{A}\text{-}\mathcal{B})$.

1) M is $\mathcal{A}$-h-projective $\Rightarrow$ $(-)\otimes_{\mathcal{A}} M$ preserves acyclic complexes & thus computes $(-)\overset{L}{\otimes}_{\mathcal{A}} M$.
2) M is $\mathcal{B}$-h-projective $\Rightarrow$ $\mathrm{Hom}_{\mathcal{B}}(M,-)$ —— " —— —— " —— $R\mathrm{Hom}_{\mathcal{B}}(M,-)$
3) k is a field, M is h-projective $\Rightarrow$ M is $\mathcal{A}$-, $\mathcal{B}$-h-projective.

Thus can compute $(-)\overset{L}{\otimes}_{\mathcal{A}} M$ & $R\mathrm{Hom}_{\mathcal{B}}(M,-)$ by taking an h-projective res-n of M.

Duals and adjoints

Definition 15: Let $\mathcal{A}$ be a DG-category. The **diagonal bimodule $\mathcal{A}$** is defined by ${}_a\mathcal{A}_b = \mathrm{Hom}_{\mathcal{A}}(b,a)$.

NB: Its fibers $\mathcal{A}_a$ & ${}_a\mathcal{A}$ are the left & right representable modules $\mathrm{Hom}_{\mathcal{A}}(a,-)$ & $\mathrm{Hom}_{\mathcal{A}}(-,a)$ coinciding with previous notation.

Exercise 16: Let $E \in \mathrm{Mod}\,\mathcal{A}$. $\mathrm{Hom}_{\mathcal{A}}({}_a\mathcal{A}, E) = E_a = E \otimes_{\mathcal{A}} \mathcal{A}_a$

$$\therefore \mathrm{Hom}_{\mathcal{A}}(\mathcal{A},-) = \mathrm{Id}_{\mathcal{A}} = (-)\otimes_{\mathcal{A}} \mathcal{A}$$

On the other hand:

Definition 17: The **dualising functor** $(-)^{\mathcal{A}} : (\mathrm{Mod}\text{-}\mathcal{A})^{\mathrm{opp}} \longrightarrow \mathrm{Mod}\text{-}\mathcal{A}^{\mathrm{opp}}$ is defined to be $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{A})$

$(-)^{\bar{\mathcal{A}}}$ denoted derived version. $a \mapsto \mathrm{Hom}_{\mathcal{A}}(-, {}_a\mathcal{A})$

Question: When is this an equivalence?

Exercise 18: a) $(-)^{\mathcal{A}}$ restricts to identity on $\mathcal{A}^{\mathrm{op}} \hookrightarrow (\mathrm{Mod}\,\mathcal{A})^{\mathrm{op}}$ & $\mathcal{A}^{\mathrm{op}} \hookrightarrow \mathrm{Mod}\,\mathcal{A}^{\mathrm{op}}$

b) $(-)^{\mathcal{A}}$ induces an equivalence $(\mathrm{SF}_{\mathrm{fg}}\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{SF}_{\mathrm{fg}}\mathcal{A}^{\mathrm{op}}$

Definition 19: Let $E \in \operatorname{Mod}\mathcal{A}$. E is a PERFECT module if it is a compact object of $D(\mathcal{A})$, i.e. $\operatorname{Hom}_{D(\mathcal{A})}(E,-)$ commutes with infinite direct sums.
Denote by $\operatorname{Perf}\mathcal{A}$ & $P^{\text{Perf}}(\mathcal{A})$ the corresponding full subcategories of $\operatorname{Mod}\mathcal{A}$ & $P(\mathcal{A})$.
Denote by $D_c(\mathcal{A})$ the full subcategory of compact objects in $D(\mathcal{A})$.

Fact: $H^0(SF_{fg}(\mathcal{A}))$ is the triangulated hull of $H^0(\mathcal{A})$ in $H^0(\operatorname{Mod}\mathcal{A})$. We denote it by $\operatorname{Tr}(\mathcal{A})$.
$D_c(\mathcal{A})$ is the Karoubi completion of $\operatorname{Tr}(\mathcal{A})$ in $D(\mathcal{A})$.
$\therefore$ $P^{\text{Perf}}(\mathcal{A})$ are homotopy retracts of the elements of $SF_{fg}(\mathcal{A})$.
$E \xrightarrow{\alpha} F$ is a (homotopy) retract if $\exists F \xrightarrow{\beta} E$ such that $\beta \circ \alpha$ is (homotopic to) Id_E

Definition 20: Let $\mathcal{A}, \mathcal{B}$ be DG categories. A functor $\mathcal{A} \xrightarrow{\Phi} \mathcal{B}$ is a QUASI-EQUIVALENCE if
- $\forall a,b \in \mathcal{A}$ $\operatorname{Hom}_{\mathcal{A}}(a,b) \to \operatorname{Hom}_{\mathcal{B}}(\Phi a, \Phi b)$ is a quasi-iso.
- $H^0(\mathcal{A}) \xrightarrow{\Phi} H^0(\mathcal{B})$ is essentially surjective $\leftarrow$ and thus an equivalence

Proposition 21: The dualising functor $(-)^{\mathcal{A}}$ restricts to a quasi-equivalence

$$(P^{\text{Perf}}(\mathcal{A}))^{op} \xrightarrow{\sim} P^{\text{Perf}}(\mathcal{A}^{op})$$

Proof: Consider the natural transformation

$$(\dagger) \qquad \mathrm{Id}_{\mathcal{A}} \longrightarrow (-)^{\mathcal{A}\mathcal{A}} = \operatorname{Hom}_{\mathcal{A}}(\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{A}),\mathcal{A})$$

defined by morphisms $M \to \operatorname{Hom}_{\mathcal{A}}(\operatorname{Hom}_{\mathcal{A}}(M,\mathcal{A}),\mathcal{A})$ adjoint to adjunction counits $M \otimes \operatorname{Hom}(M,\mathcal{A}) \to \mathcal{A}$.
$\forall a \in \mathcal{A}$ $({}_a\mathcal{A})^{\mathcal{A}} = \mathcal{A}_a$, thus $(\dagger)$ is the identity map on $\mathcal{A}^{op}$ in $(\operatorname{Mod}\mathcal{A})^{op}$ & $\operatorname{Mod}\mathcal{A}^{op}$.
Since $SF_{fg}(\mathcal{A}^{op})$ is the triangulated hull of $\mathcal{A}^{op}$ $\Rightarrow$ $(\dagger)$ is a homotopy equivalence on $SF(\mathcal{A}^{op})$
Similarly, since all retracts in a triangulated category are direct summands $\Rightarrow$
$\Rightarrow$ $(\dagger)$ is a homotopy equivalence on $P^{\text{Perf}}(\mathcal{A}^{op})$ & claim follows. $\square$

Let $C, D \in \operatorname{Mod}\mathcal{A}$. We have a natural map

$$C \otimes_{\mathcal{A}} D^{\mathcal{A}} \longrightarrow \operatorname{Hom}_{\mathcal{A}}(D,C) \qquad \operatorname{Hom}_{\mathcal{A}}(\mathcal{A},C) \otimes_{\mathcal{A}} \operatorname{Hom}_{\mathcal{A}}(D,\mathcal{A}) \xrightarrow{\text{comp.}} \operatorname{Hom}_{\mathcal{A}}(D,C)$$

Same argument as above $\Rightarrow$ it is a homotopy equivalence if either C or D are perfect.

Proposition 22: Let $M \in \operatorname{Mod}(\mathcal{A}\text{-}\mathcal{B})$.
a) M is $\mathcal{B}$-h-projective & $\mathcal{B}$-perfect $\Rightarrow$ $(-)\otimes_{\mathcal{B}} M^{\mathcal{B}}$ is a homotopy right adjoint of $(-)\otimes_{\mathcal{A}} M$.
b) M is $\mathcal{A}$-h-projective & $\mathcal{A}$-perfect $\Rightarrow$ $(-)\otimes_{\mathcal{B}} M^{\mathcal{A}}$ is a homotopy left adjoint of $(-)\otimes_{\mathcal{A}} M$.

Corollary 23: Let $M \in \operatorname{Mod}(\mathcal{A}\text{-}\mathcal{B})$.
a) If M is $\mathcal{B}$-perfect, $(-)\overset{L}{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{B}}}$ is the right adjoint of $(-)\overset{L}{\otimes}_{\mathcal{A}} M$.
b) If M is $\mathcal{A}$-perfect, $(-)\overset{L}{\otimes}_{\mathcal{B}} M^{\mathcal{A}}$ is the left adjoint of $(-)\overset{L}{\otimes}_{\mathcal{A}} M$.

Adj. co-unit: $M^{\bar{\mathcal{B}}} \overset{L}{\otimes}_{\mathcal{A}} M \xrightarrow{ev} \mathcal{B}$
Adj. unit: $\mathcal{A} \xrightarrow{act} R\operatorname{Hom}_{\mathcal{B}}(M,M) \overset{?!}{\simeq} M \overset{L}{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{B}}}$

PRE-TRIANGULATED CATEGORIES AND TWISTED COMPLEXES

RECALL: $H^0(\mathcal{A}) \hookrightarrow H^0(\mathrm{Mod}\,\mathcal{A})$ ⟵ NATURALLY TRIANGULATED.

DEFINITION 24: A DG-CATEGORY $\mathcal{A}$ is **PRE-TRIANGULATED** IF $H^0(\mathcal{A})$ IS A TRIANGULATED SUBCATEGORY OF $H^0(\mathrm{Mod}\text{-}\mathcal{A})$.